Macroeconomics I: Macroeconomic Principles
Effectiveness of policy and time inconsistency

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Class of October 27, 2008

1 The (in)ability of policy to alter output

1.1 The timing of monetary policy

We focus on rational expectations. The model consist of an AS relation, an AD relation and a rule for monetary policy:

$$\begin{aligned} y_t &= y^n + a\left(p_t - E_{t-1}p_t\right) + u_t \\ y_t &= m_t - p_t + v_t \\ m_t &= -\varphi_1 y_t - \varphi_2 y_{t-1} + \varepsilon_t \end{aligned}$$

where u_t, v_t and ε_t are shocks of zero expected value. $E_{t-1}p_t$ is the expectation of the price level based on information at time $t-1$.

Substitute the rule for monetary policy in the AD relation:

$$\begin{aligned} y_t &= m_t - p_t + v_t \\ y_t &= -\varphi_1 y_t - \varphi_2 y_{t-1} + \varepsilon_t - p_t + v_t \\ y_t &= -\frac{\varphi_2}{1+\varphi_1} y_{t-1} - \frac{1}{1+\varphi_1} p_t + \frac{\varepsilon_t + v_t}{1+\varphi_1} \end{aligned} \tag{1}$$

Use (1) to substitute for output in the AS relation:

$$-\frac{\varphi_2}{1+\varphi_1} y_{t-1} - \frac{1}{1+\varphi_1} p_t + \frac{\varepsilon_t + v_t}{1+\varphi_1} = y^* + a\left(p_t - E_{t-1}p_t\right) + u_t \tag{2}$$

Take the expectations of both sides at time $t-1$:

$$\begin{aligned}
-\frac{\varphi_2}{1+\varphi_1}y_{t-1} - \frac{1}{1+\varphi_1}E_{t-1}p_t + \frac{E_{t-1}\varepsilon_t + E_{t-1}v_t}{1+\varphi_1} &= y^n + a\left(E_{t-1}p_t - E_{t-1}p_t\right) + E_{t-1}u_t \\
-\frac{\varphi_2}{1+\varphi_1}y_{t-1} - \frac{1}{1+\varphi_1}E_{t-1}p_t &= y^n
\end{aligned}$$

which implies:

$$E_{t-1}p_t = -\varphi_2 y_{t-1} - (1+\varphi_1)\, y^n \tag{3}$$

Now use (3) to substitute for the expected price in (2):

$$\begin{aligned}
-\frac{\varphi_2}{1+\varphi_1}y_{t-1} - \frac{1}{1+\varphi_1}p_t + \frac{\varepsilon_t + v_t}{1+\varphi_1} &= y^n + a\left(p_t - E_{t-1}p_t\right) + u_t \\
-\frac{\varphi_2}{1+\varphi_1}y_{t-1} - \frac{1}{1+\varphi_1}p_t + \frac{\varepsilon_t + v_t}{1+\varphi_1} &= y^n + a\left(p_t + \varphi_2 y_{t-1} + (1+\varphi_1)\, y^*\right) + u_t \\
-\left[\frac{1}{1+\varphi_1} + a\right]\varphi_2 y_{t-1} - \left[\frac{1}{1+\varphi_1} + a\right]p_t &= \left[1 + a\,(1+\varphi_1)\right] y^n + u_t - \frac{\varepsilon_t + v_t}{1+\varphi_1}
\end{aligned}$$

which implies:

$$p_t = -\varphi_2 y_{t-1} + \frac{\varepsilon_t + v_t - (1+\varphi_1)\, u_t}{1 + a\,(1+\varphi_1)} - (1+\varphi_1)\, y^n \tag{4}$$

Using (3) and (4) in the AS relation we solve for output:

$$\begin{aligned}
y_t &= y^n + a\left(p_t - E_{t-1}p_t\right) + u_t \\
y_t &= y^n + a\frac{\varepsilon_t + v_t - (1+\varphi_1)\, u_t}{1 + a\,(1+\varphi_1)} + u_t \\
y_t &= y^n + \frac{a\left(\varepsilon_t + v_t\right) + u_t}{1 + a\,(1+\varphi_1)} \qquad (5)
\end{aligned}$$

1.2 Wage staggering

The model consist of an AS relation where output is inversely related to the real wage, an AD relation and a rule for monetary policy:

$$\begin{aligned}
y_t &= y^n + a\left(p_t - w_t\right) + e_t \\
y_t &= m_t - p_t + v_t \\
m_t &= \sum_{i=1}^{\infty}\alpha_i e_{t-i} + \sum_{i=1}^{\infty}\beta_i v_{t-i}
\end{aligned}$$

Monetary policy can only react to lagged shocks, not current ones. In particular:

$$\begin{aligned}
E_{t-1}m_t &= m_t \\
E_{t-2}m_t &= \alpha_1 E_{t-2}e_{t-1} + \beta_1 E_{t-2}v_{t-1} + \sum_{i=2}^{\infty}\alpha_i e_{t-i} + \sum_{i=2}^{\infty}\beta_i v_{t-i} \\
&= \sum_{i=2}^{\infty}\alpha_i e_{t-i} + \sum_{i=2}^{\infty}\beta_i v_{t-i} \\
m_t - E_{t-2}m_t &= \alpha_1 e_{t-1} + \beta_1 v_{t-1}
\end{aligned}$$

Half the wages are reset each period, based on expectations for prices. The wage setting is then:

$$w_t = \frac{1}{2}E_{t-1}p_t + \frac{1}{2}E_{t-2}p_t$$

We combine AS and AD to substitute for output:

$$\begin{aligned}
m_t - p_t + v_t &= y^n + a\left(p_t - w_t\right) + e_t \\
m_t - p_t + v_t &= y^n + a\left(p_t - \frac{1}{2}E_{t-1}p_t + \frac{1}{2}E_{t-2}p_t\right) + e_t \qquad (6)
\end{aligned}$$

Take the expectation of (6) from the point of view of period $t-2$:

$$\begin{aligned}
E_{t-2}m_t - E_{t-2}p_t + E_{t-2}v_t &= E_{t-2}y^n + a\left(E_{t-2}p_t - \frac{1}{2}E_{t-2}\left(E_{t-1}p_t\right) + \frac{1}{2}E_{t-2}p_t\right) + E_{t-2}e_t \\
E_{t-2}m_t - E_{t-2}p_t &= y^n + a\left(1 - \frac{1}{2} + \frac{1}{2}\right)E_{t-2}p_t \\
E_{t-2}m_t - E_{t-2}p_t &= y^n \\
E_{t-2}p_t &= E_{t-2}m_t - y^n \qquad (7)
\end{aligned}$$

Take the expectation of (6) from the point of view of period $t-1$:

$$\begin{aligned}
E_{t-1}m_t - E_{t-1}p_t + E_{t-1}v_t &= E_{t-1}y^n + a\left(E_{t-1}p_t - \frac{1}{2}E_{t-1}p_t - \frac{1}{2}E_{t-2}p_t\right) + E_{t-1}e_t \\
E_{t-1}m_t - E_{t-1}p_t &= y^n + a\left(\frac{1}{2}E_{t-1}p_t - \frac{1}{2}E_{t-2}p_t\right) \\
E_{t-1}m_t &= y^n + \frac{2+a}{2}E_{t-1}p_t - \frac{a}{2}E_{t-2}p_t \\
E_{t-1}p_t &= -\frac{2}{2+a}y^n + \frac{2}{2+a}E_{t-1}m_t + \frac{a}{2+a}E_{t-2}p_t \qquad (8) \\
E_{t-1}p_t &= -y^n + \frac{2}{2+a}E_{t-1}m_t + \frac{a}{2+a}E_{t-2}m_t
\end{aligned}$$

Use (7) and (8) in (6):

$$\begin{aligned}
m_t - p_t + v_t &= y^n + a\left(p_t - \frac{1}{2}E_{t-1}p_t - \frac{1}{2}E_{t-2}p_t\right) + e_t \\
m_t - p_t + v_t &= y^n + ap_t \\
&\quad -\frac{a}{2}\left[-\frac{2}{2+a}y^* + \frac{2}{2+a}E_{t-1}m_t + \frac{a}{2+a}E_{t-2}p_t\right] - \frac{a}{2}E_{t-2}p_t + e_t \\
m_t - p_t + v_t &= y^n + a\left(p_t + \frac{1}{2+a}y^* - \frac{1}{2+a}E_{t-1}m_t - \frac{1+a}{2+a}E_{t-2}p_t\right) + e_t \\
m_t - p_t + v_t &= y^n + ap_t + \frac{a}{2+a}y^* - \frac{a}{2+a}E_{t-1}m_t - \frac{1+a}{2+a}aE_{t-2}p_t + e_t \\
m_t - p_t + v_t &= y^n + ap_t + \frac{a}{2+a}y^* - \frac{a}{2+a}E_{t-1}m_t - \frac{1+a}{2+a}a\left[E_{t-2}m_t - y^*\right] + e_t \\
m_t - (1+a)\,p_t + v_t &= \left[1 + \frac{a}{2+a} + \frac{1+a}{2+a}a\right]y^n \\
&\quad -\frac{a}{2+a}E_{t-1}m_t - \frac{1+a}{2+a}aE_{t-2}m_t + e_t \\
m_t - (1+a)\,p_t + v_t &= (1+a)\,y^n - \frac{a}{2+a}E_{t-1}m_t - \frac{1+a}{2+a}aE_{t-2}m_t + e_t \\
(1+a)\,p_t &= m_t + v_t - (1+a)\,y^n + \frac{a}{2+a}E_{t-1}m_t + \frac{1+a}{2+a}aE_{t-2}m_t - e_t \\
(1+a)\,p_t &= v_t - (1+a)\,y^n + 2\frac{1+a}{2+a}m_t + \frac{1+a}{2+a}aE_{t-2}m_t - e_t \\
p_t &= \frac{2}{2+a}m_t + \frac{a}{2+a}E_{t-2}m_t + \frac{v_t - e_t}{1+a} - y^n \qquad (9)
\end{aligned}$$

Using (7), (8) and (9) AS becomes:

$$\begin{aligned}
y_t &= y^n + a\left(p_t - \frac{1}{2}E_{t-1}p_t - \frac{1}{2}E_{t-2}p_t\right) + e_t \\
y_t &= y^n + a\left(p_t - \frac{1}{2}\left[-y^n + \frac{2}{2+a}E_{t-1}m_t + \frac{a}{2+a}E_{t-2}m_t\right] - \frac{1}{2}\left[E_{t-2}m_t - y^n\right]\right) + e_t \\
y_t &= y^n + a\left(y^n + p_t - \frac{1}{2+a}E_{t-1}m_t - \frac{1+a}{2+a}E_{t-2}m_t\right) + e_t \\
y_t &= (1+a)\,y^n + ap_t - \frac{a}{2+a}E_{t-1}m_t - \frac{1+a}{2+a}aE_{t-2}m_t + e_t \\
y_t &= (1+a)\,y^n + a\left(\frac{2}{2+a}m_t + \frac{a}{2+a}E_{t-2}m_t + \frac{v_t - e_t}{1+a} - y^n\right) \\
&\quad -\frac{a}{2+a}E_{t-1}m_t - \frac{1+a}{2+a}aE_{t-2}m_t + e_t \\
y_t &= y^n + \frac{a}{2+a}m_t - \frac{a}{2+a}E_{t-2}m_t + a\frac{v_t - e_t}{1+a} + e_t \\
y_t &= y^n + \frac{a}{2+a}\left[m_t - E_{t-2}m_t\right] + \frac{av_t + e_t}{1+a} \qquad (10) \\
y_t &= y^n + \frac{a}{2+a}\left[\alpha_1 u_{t-1} + \beta_1 v_{t-1}\right] + \frac{av_t + e_t}{1+a}
\end{aligned}$$

2 Time inconsistency in monetary policy

2.1 General setup

The objective of the central bank is to minimize a quadratic loss function in output and inflation:

$$L = \frac{1}{2}\left[\lambda\left(y - y^*\right)^2 + \left(\pi - \pi^*\right)^2\right]$$

where y^* and π^* are the ideal output and inflation. For simplicity set $\pi^* = 0$.

The economy is characterized by an AS relation (π^e is expected inflation):

$$y = y^n + a\left(\pi - \pi^e\right) + e$$

The gap between the ideal and natural output is:

$$k = y^* - y^n > 0$$

Inflation is set by the central bank.

2.2 Equilibrium under discretion

The central bank takes π^e as given and minimizes:

$$\begin{aligned} L &= \frac{1}{2}\left[\lambda\left(y - y^n - k\right)^2 + \pi^2\right] \\ &= \frac{1}{2}\left[\lambda\left(a\left(\pi - \pi^e\right) + e - k\right)^2 + \pi^2\right] \end{aligned}$$

The optimality condition is (d denotes discretion):

$$\begin{aligned} 0 &= \frac{\partial L}{\partial \pi} = \lambda\left(a\left(\pi^d - \pi^{ed}\right) + e - k\right)a + \pi^d \\ 0 &= -\lambda a^2 \pi^{ed} + a\lambda e - a\lambda k + \left(1 + a^2\lambda\right)\pi^d \\ \pi^d &= \frac{\lambda a^2}{1 + a^2\lambda}\pi^{ed} + \frac{a\lambda}{1 + a^2\lambda}\left(k - e\right) \end{aligned}$$

Take expectations on both sides:

$$\begin{aligned} \pi^{ed} &= \frac{\lambda a^2}{1 + a^2\lambda}\pi^{ed} + \frac{a\lambda}{1 + a^2\lambda}\left(k - e^e\right) \\ \pi^{ed} &= \frac{\lambda a^2}{1 + a^2\lambda}\pi^{ed} + \frac{a\lambda}{1 + a^2\lambda}k \\ \left(1 + a^2\lambda\right)\pi^{ed} &= \lambda a^2 \pi^{ed} + a\lambda k \\ \pi^{ed} &= a\lambda k \end{aligned} \tag{11}$$

hence:

$$\begin{aligned} \pi^d &= \frac{\lambda a^2}{1 + a^2\lambda}\pi^{ed} + \frac{a\lambda}{1 + a^2\lambda}\left(k - e\right) \\ \pi^d &= \frac{\lambda a^2}{1 + a^2\lambda}a\lambda k + \frac{a\lambda}{1 + a^2\lambda}\left(k - e\right) \\ \pi^d &= a\lambda k - \frac{a\lambda}{1 + a^2\lambda}e \end{aligned} \tag{12}$$

Output is given by AS:

$$\begin{aligned} y^d &= y^n + a\left(\pi^d - \pi^{ed}\right) + e \\ &= y^n + a\left(-\frac{a\lambda}{1 + a^2\lambda}e\right) + e \\ &= y^n + \frac{1}{1 + a^2\lambda}e \end{aligned} \tag{13}$$

The expected loss is then:

$$\begin{aligned}
EL^d &= \frac{1}{2}E\left[\lambda\left(a\left(\pi^d-\pi^{ed}\right)+e-k\right)^2+\left(\pi^d\right)^2\right] \\
&= \frac{1}{2}E\left[\lambda\left(\frac{1}{1+a^2\lambda}e-k\right)^2+\left(a\lambda k-\frac{a\lambda}{1+a^2\lambda}e\right)^2\right] \\
&= \frac{1}{2}E\left[\begin{array}{c}\lambda\left(\frac{1}{1+a^2\lambda}e\right)^2+\lambda\left(k\right)^2-2\lambda\frac{1}{1+a^2\lambda}ek \\ +\left(\frac{a\lambda}{1+a^2\lambda}e\right)^2+\left(a\lambda k\right)^2-2a\lambda k\frac{a\lambda}{1+a^2\lambda}e\end{array}\right] \\
&= \frac{1}{2}E\left[\frac{\lambda+a^2\lambda^2}{\left(1+a^2\lambda\right)^2}\left(e\right)^2+\left(\lambda+a^2\lambda^2\right)\left(k\right)^2-2\frac{\left(\lambda+a^2\lambda^2\right)}{1+a^2\lambda}ek\right] \\
&= \frac{1}{2}E\left[\frac{\lambda}{1+a^2\lambda}\left(e\right)^2+\lambda\left(1+a^2\lambda\right)\left(k\right)^2-2\lambda ek\right] \\
&= \frac{1}{2}\left[\frac{\lambda}{1+a^2\lambda}E\left(e\right)^2+\lambda\left(1+a^2\lambda\right)\left(k\right)^2-2\lambda kEe\right] \\
&= \frac{1}{2}\left[\frac{\lambda}{1+a^2\lambda}\left(E\left(e\right)^2-\left[E\left(e\right)\right]^2\right)+\lambda\left(1+a^2\lambda\right)\left(k\right)^2\right] \\
&= \frac{1}{2}\frac{\lambda}{1+a^2\lambda}\sigma_e^2+\frac{\lambda}{2}\left(1+a^2\lambda\right)\left(k\right)^2 \qquad (14)
\end{aligned}$$

2.3 Equilibrium under commitment

The central bank announces a rule for inflation and sticks to it:

$$\pi^c = b_0 + b_1 e$$

where c denotes commitment, and b_0 and b_1 are coefficients that need to be determined. Expected inflation is then:

$$\pi^{ec} = b_0$$

Output is given by AS:

$$\begin{aligned}
y^c &= y^n + a\left(\pi^c-\pi^{ec}\right)+e \\
&= y^n + ab_1 e + e \\
&= y^n + \left(1+ab_1\right)e
\end{aligned}$$

The expected loss is then:

$$\begin{aligned} EL^c &= \frac{1}{2}E\left[\lambda\left(a\left(\pi^c-\pi^{ec}\right)+e-k\right)^2+\left(\pi^c\right)^2\right] \\ &= \frac{1}{2}E\left[\lambda\left(\left(1+ab_1\right)-k\right)^2+\left(b_0+b_1e\right)^2\right] \end{aligned}$$

Set the derivative of this expression with respect to b_0 to zero:

$$\begin{aligned} 0 &= \frac{\partial EL^c}{\partial b_0} = E\left(b_0+b_1e\right) \\ &= b_0+b_1Ee \\ &= b_0 \end{aligned}$$

Therefore $b_0=0$ and the expected loss is:

$$EL^c = \frac{1}{2}E\left[\lambda\left(\left(1+ab_1\right)e-k\right)^2+\left(b_1e\right)^2\right]$$

Set the derivative of this expression with respect to b_1 to zero:

$$\begin{aligned} 0 &= \frac{\partial EL^c}{\partial b_1} = E\left[\lambda\left(\left(1+ab_1\right)e-k\right)ae+\left(b_1e\right)e\right] \\ &= \lambda\left(\left(1+ab_1\right)e-k\right)ae+b_1E\left(e\right)^2 \\ &= \lambda\left(1+ab_1\right)aE\left(e\right)^2-ka\lambda^2E\left(e\right)+b_1E\left(e\right)^2 \\ &= \left[\left(1+ab_1\right)a\lambda+b_1\right]E\left(e\right)^2 \\ &= \left(1+ab_1\right)a\lambda+b_1 \\ &= a\lambda+\left(1+a^2\lambda\right)b_1 \\ b_1 &= -\frac{a\lambda}{1+a^2\lambda} \end{aligned}$$

Inflation and output are then:

$$\pi^c = b_0+b_1e = -\frac{a\lambda}{1+a^2\lambda}e \tag{15}$$

$$\pi^{ec} = 0 \tag{16}$$

$$y^c = y^n+\frac{1}{1+a^2\lambda}e \tag{17}$$

The expected loss is:

$$\begin{aligned}
EL^c &= \frac{1}{2}E\left[\lambda\left(\frac{1}{1+a^2\lambda}e-k\right)^2+\left(-\frac{a\lambda}{1+a^2\lambda}e\right)^2\right] \\
&= \frac{1}{2}E\left[\lambda\left(\frac{1}{1+a^2\lambda}e\right)^2+\lambda(k)^2-2\lambda\frac{1}{1+a^2\lambda}ek+\left(\frac{a\lambda}{1+a^2\lambda}e\right)^2\right] \\
&= \frac{1}{2}\left[\lambda\left(\frac{1}{1+a^2\lambda}\right)^2E(e)^2+\lambda(k)^2-2\lambda\frac{1}{1+a^2\lambda}kE(e)+\left(\frac{a\lambda}{1+a^2\lambda}\right)^2E(e)^2\right] \\
&= \frac{1}{2}\left[\lambda\left(\frac{1}{1+a^2\lambda}\right)^2\sigma_e^2+\lambda(k)^2+\left(\frac{a\lambda}{1+a^2\lambda}\right)^2\sigma_e^2\right] \\
&= \frac{1}{2}\left[\lambda+(a\lambda)^2\right]\left(\frac{1}{1+a^2\lambda}\right)^2\sigma_e^2+\frac{\lambda}{2}(k)^2 \\
&= \frac{1}{2}\frac{\lambda}{1+a^2\lambda}\sigma_e^2+\frac{\lambda}{2}(k)^2 \qquad (18)
\end{aligned}$$

2.4 Equilibrium under strict commitment

The central bank announces a narrow rule for inflation that excludes reacting to shocks:

$$\pi^{sc}=b_0$$

Following the same steps as above we get $b_0=0$, hence inflation and output are :

$$\begin{aligned}
\pi^{sc} &= 0 \\
y^{sc} &= y^n+a(\pi^{sc}-\pi^{esc})+e=y^n+e
\end{aligned}$$

The expected loss is:

$$\begin{aligned}
EL^{sc} &= \frac{1}{2}E\left[\lambda(e-k)^2\right] \\
&= \frac{1}{2}\left[\lambda E(e)^2+\lambda(k)^2-2\lambda kEe\right] \\
&= \frac{\lambda}{2}\sigma_e^2+\frac{\lambda}{2}(k)^2 \qquad (19)
\end{aligned}$$

2.5 Delegation

Monetary policy is delegated to a central bank whose loss function is:

$$L = \frac{1}{2}\left[\lambda\left(y - y^n - k\right)^2 + \left(1+\delta\right)\left(\pi\right)^2\right]$$

The central banker cares more about inflation than society does ($\delta > 1$). The banker conducts policy under discretion, so the solution is as above, except that λ is replaced by $\lambda/\left(1+\delta\right)$. (12) and (13) are then:

$$\begin{aligned}
\pi^d &= \frac{a\lambda}{1+\delta}k - \frac{a\frac{\lambda}{1+\delta}}{1+a^2\frac{\lambda}{1+\delta}}e = \frac{a\lambda}{1+\delta}k - \frac{a\lambda}{1+\delta+a^2\lambda}e \\
y^d &= y^n + \frac{1}{1+a^2\frac{\lambda}{1+\delta}}e = y^n + \frac{1+\delta}{1+\delta+a^2\lambda}e
\end{aligned}$$

The expected loss for society is then:

$$\begin{aligned}
EL &= \frac{1}{2}\left[\lambda E\left(y - y^n - k\right)^2 + E\left(\pi\right)^2\right] \\
&= \frac{1}{2}\left[\lambda E\left(\frac{1+\delta}{1+\delta+a^2\lambda}e - k\right)^2 + E\left(\frac{a\lambda}{1+\delta}k - \frac{a\lambda}{1+\delta+a^2\lambda}e\right)^2\right] \\
&= \frac{1}{2}\left[\begin{array}{c}\lambda\left(\frac{1+\delta}{1+\delta+a^2\lambda}\right)^2 E\left(e\right)^2 + \lambda\left(k\right)^2 - 2\lambda\frac{1+\delta}{1+\delta+a^2\lambda}kEe \\ +\left(\frac{a\lambda}{1+\delta+a^2\lambda}\right)^2 E\left(e\right)^2 + \left(\frac{a\lambda}{1+\delta}k\right)^2 - 2\frac{a\lambda}{1+\delta}\frac{a\lambda}{1+\delta+a^2\lambda}kEe\end{array}\right] \\
&= \frac{1}{2}\left[\lambda\left[\left(1+\delta\right)^2 + a^2\lambda\right]\left(\frac{1}{1+\delta+a^2\lambda}\right)^2 E\left(e\right)^2 + \lambda\left(k\right)^2\left[1 + \left(\frac{a}{1+\delta}\right)^2\lambda\right]\right]
\end{aligned}$$

The optimal extent of delegation sets δ to minimize this:

$$\begin{aligned}
0 &= \frac{\partial EL}{\partial \delta} = \frac{1}{2}\left[\begin{array}{c} 2\lambda\left(1+\delta\right)\left(\frac{1}{1+\delta+a^2\lambda}\right)^2 E\left(e\right)^2 \\ -2\lambda\left[\left(1+\delta\right)^2+a^2\lambda\right]\left(\frac{1}{1+\delta+a^2\lambda}\right)^3 E\left(e\right)^2 \\ -2\lambda\left(k\right)^2\left(\frac{1}{1+\delta}\right)^3 a^2\lambda \end{array}\right] \\
0 &= \lambda\left(1+\delta\right)\left(\frac{1}{1+\delta+a^2\lambda}\right)^2 E\left(e\right)^2 \\
&\quad -\lambda\left[\left(1+\delta\right)^2+a^2\lambda\right]\left(\frac{1}{1+\delta+a^2\lambda}\right)^3 E\left(e\right)^2 - \lambda\left(k\right)^2\left(\frac{1}{1+\delta}\right)^3 a^2\lambda \\
0 &= \left[\left(1+\delta\right)\left(1+\delta+a^2\lambda\right)-\left[\left(1+\delta\right)^2+a^2\lambda\right]\right]\left(\frac{1}{1+\delta+a^2\lambda}\right)^3\sigma_e^2 \\
&\quad -\lambda\left(k\right)^2\left(\frac{1}{1+\delta}\right)^3 a^2 \\
0 &= \left[\left(1+\delta\right)a^2\lambda - a^2\lambda\right]\left(\frac{1}{1+\delta+a^2\lambda}\right)^3\sigma_e^2 - \lambda\left(k\right)^2\left(\frac{1}{1+\delta}\right)^3 a^2 \\
0 &= \delta a^2\lambda\left(\frac{1}{1+\delta+a^2\lambda}\right)^3\sigma_e^2 - \lambda\left(k\right)^2\left(\frac{1}{1+\delta}\right)^3 a^2 \\
0 &= \delta a^2\lambda\left(\frac{1+\delta}{1+\delta+a^2\lambda}\right)^3\sigma_e^2 - \lambda\left(k\right)^2 a^2 \\
0 &= \delta\left(\frac{1+\delta}{1+\delta+a^2\lambda}\right)^3\sigma_e^2 - \left(k\right)^2 \\
\delta &= \frac{\left(k\right)^2}{\sigma_e^2}\left(\frac{1+\delta+a^2\lambda}{1+\delta}\right)^3 = g\left(\delta\right) \qquad (20)
\end{aligned}$$

Notice that:

$$\begin{aligned}
g'\left(\delta\right) &= \frac{\left(k\right)^2}{\sigma_e^2}3\left(\frac{1+\delta+a^2\lambda}{1+\delta}\right)^2\frac{\left(1+\delta\right)-\left(1+\delta+a^2\lambda\right)}{\left(1+\delta\right)^2} \\
&= \frac{\left(k\right)^2}{\sigma_e^2}3\left(\frac{1+\delta+a^2\lambda}{1+\delta}\right)^2\frac{-a^2\lambda}{\left(1+\delta\right)^2} < 0 \\
g\left(0\right) &= \frac{\left(k\right)^2}{\sigma_e^2}\left(1+a^2\lambda\right)^3 > 0 \\
g\left(+\infty\right) &= \frac{\left(k\right)^2}{\sigma_e^2} < \infty
\end{aligned}$$

Therefore (20) implies a positive value of δ. In addition, we can differentiate (20) as follows:

$$\begin{aligned} d\delta &= \frac{2k}{\sigma_e^2}\left(\frac{1+\delta+a^2\lambda}{1+\delta}\right)^3 dk - \frac{(k)^2}{\sigma_e^2}\left(\frac{1+\delta+a^2\lambda}{1+\delta}\right)^3 \frac{d\sigma_e^2}{\sigma_e^2} \\ &+\frac{(k)^2}{\sigma_e^2}3\left(\frac{1+\delta+a^2\lambda}{1+\delta}\right)^2 \frac{2a\lambda}{1+\delta}da \\ &+\frac{(k)^2}{\sigma_e^2}3\left(\frac{1+\delta+a^2\lambda}{1+\delta}\right)^2 \frac{a^2}{1+\delta}d\lambda \\ &+\frac{(k)^2}{\sigma_e^2}3\left(\frac{1+\delta+a^2\lambda}{1+\delta}\right)^2 \frac{-(a^2\lambda)}{(1+\delta)^2}d\delta \end{aligned}$$

which implies:

$$\begin{aligned} &\left(1+3\frac{\delta}{1+\delta}\frac{a^2\lambda}{1+\delta+a^2\lambda}\right)d\delta \\ &= 2\delta\frac{dk}{k} - \delta\frac{d\sigma_e^2}{\sigma_e^2} + \frac{3\delta a^2\lambda}{1+\delta+a^2\lambda}\left(2\frac{da}{a}+\frac{d\lambda}{\lambda}\right) \end{aligned}$$

Note that the variance of output and inflation are:

$$\begin{aligned} Var\left(\pi^d\right) &= \left(\frac{a\lambda}{1+\delta+a^2\lambda}\right)^2\sigma_e^2 < \left(\frac{a\lambda}{1+a^2\lambda}\right)^2\sigma_e^2 \\ Var\left(y^d\right) &= \left(\frac{1+\delta}{1+\delta+a^2\lambda}\right)^2\sigma_e^2 > \left(\frac{1}{1+a^2\lambda}\right)^2\sigma_e^2 \end{aligned}$$

2.6 Optimal contract

The central banker maximizes a function that includes a payoff in addition to the social loss:

$$\begin{aligned} U &= (t_0 - t_1\pi) - \frac{1}{2}\left[\lambda\left(y-y^n-k\right)^2+(\pi)^2\right] \\ &= (t_0 - t_1\pi) - \frac{1}{2}\left[\lambda\left(a\left(\pi-\pi^e\right)+e-k\right)^2+(\pi)^2\right] \end{aligned}$$

The central banker maximizes this payoff under discretion. The first order condition with respect to inflation is:

$$\begin{aligned} 0 &= \frac{\partial U}{\partial \pi} = -t_1 - \left[a\lambda\left(a\left(\pi - \pi^e\right) + e - k\right) + \pi\right] \\ 0 &= t_1 + a\lambda\left(a\left(\pi - \pi^e\right) + e - k\right) + \pi \\ 0 &= t_1 + a\lambda\left(-a\pi^e + e - k\right) + \left(1 + a^2\lambda\right)\pi \\ \pi &= -\frac{1}{1+a^2\lambda}t_1 + \frac{\lambda a^2}{1+a^2\lambda}\pi^e + \frac{a\lambda}{1+a^2\lambda}\left(k - e\right) \end{aligned} \tag{21}$$

Taking expectations on both sides of (21) we get:

$$\begin{aligned} \pi^e &= -\frac{1}{1+a^2\lambda}t_1 + \frac{\lambda a^2}{1+a^2\lambda}\pi^e + \frac{a\lambda}{1+a^2\lambda}k \\ \left(1 + a^2\lambda\right)\pi^e &= -t_1 + \lambda a^2\pi^e + a\lambda k \\ \pi^e &= -t_1 + a\lambda k \end{aligned} \tag{22}$$

Actual inflation is computed by combining (21) and (22):

$$\begin{aligned} \pi &= -\frac{1}{1+a^2\lambda}t_1 + \frac{\lambda a^2}{1+a^2\lambda}\left[-t_1 + a\lambda k\right] + \frac{a\lambda}{1+a^2\lambda}\left(k - e\right) \\ \pi &= -\frac{1+a^2\lambda}{1+a^2\lambda}t_1 + \frac{\lambda a^2}{1+a^2\lambda}a\lambda k + \frac{a\lambda}{1+a^2\lambda}\left(k - e\right) \\ \pi &= -t_1 + a\lambda k - \frac{a\lambda}{1+a^2\lambda}e \end{aligned} \tag{23}$$

Output is given by AS:

$$\begin{aligned} y^d &= y^n + a\left(\pi^d - \pi^{ed}\right) + e \\ &= y^n + a\left(-\frac{a\lambda}{1+a^2\lambda}e\right) + e \\ &= y^n + \frac{1}{1+a^2\lambda}e \end{aligned} \tag{24}$$

The expected social loss is then:

$$\begin{aligned}
EL &= \frac{1}{2}E\left[\lambda\left(a\left(\pi-\pi^{e}\right)+e-k\right)^{2}+\left(\pi\right)^{2}\right] \\
&= \frac{1}{2}E\left[\lambda\left(-\frac{a^{2}\lambda}{1+a^{2}\lambda}e+e-k\right)^{2}+\left(-t_{1}+a\lambda k-\frac{a\lambda}{1+a^{2}\lambda}e\right)^{2}\right] \\
&= \frac{1}{2}E\left[\lambda\left(\frac{1}{1+a^{2}\lambda}e-k\right)^{2}+\left(-t_{1}+a\lambda k-\frac{a\lambda}{1+a^{2}\lambda}e\right)^{2}\right] \\
&= \frac{1}{2}E\left[\begin{array}{c}\lambda\left(\frac{1}{1+a^{2}\lambda}e\right)^{2}+\lambda\left(k\right)^{2}-2\lambda\frac{1}{1+a^{2}\lambda}ek \\ +\left(\frac{a\lambda}{1+a^{2}\lambda}e\right)^{2}+\left(a\lambda k-t_{1}\right)^{2}-2\left(a\lambda k-t_{1}\right)\frac{a\lambda}{1+a^{2}\lambda}e\end{array}\right] \\
&= \frac{1}{2}\left[\begin{array}{c}\lambda\left(\frac{1}{1+a^{2}\lambda}\right)^{2}E\left(e\right)^{2}+\lambda\left(k\right)^{2}-2\lambda\frac{1}{1+a^{2}\lambda}kEe \\ +\left(\frac{a\lambda}{1+a^{2}\lambda}\right)^{2}E\left(e\right)^{2}+\left(a\lambda k-t_{1}\right)^{2}-2\left(a\lambda k-t_{1}\right)\frac{a\lambda}{1+a^{2}\lambda}Ee\end{array}\right] \\
&= \frac{1}{2}\left[\left(\frac{1}{1+a^{2}\lambda}\right)^{2}\left[\lambda+\left(a\lambda\right)^{2}\right]E\left(e\right)^{2}+\lambda\left(k\right)^{2}+\left(a\lambda k-t_{1}\right)^{2}\right] \\
&= \frac{1}{2}\left[\frac{\lambda}{1+a^{2}\lambda}\sigma_{e}^{2}+\lambda\left(k\right)^{2}+\left(a\lambda k-t_{1}\right)^{2}\right]
\end{aligned}$$

The optimal t_1 minimizes this loss:

$$\begin{aligned}
0 &= \frac{\partial EL}{\partial t_{1}}=-\left(a\lambda k-t_{1}\right) \\
0 &= a\lambda k-t_{1} \\
t_{1} &= a\lambda k
\end{aligned} \tag{25}$$

(23) and (22) then become:

$$\pi=-\frac{a\lambda}{1+a^{2}\lambda}e \qquad\qquad \pi^{e}=0$$

The expected social loss is then:

$$\begin{aligned}
EL &= \frac{1}{2}\left[\frac{\lambda}{1+a^{2}\lambda}\sigma_{e}^{2}+\lambda\left(k\right)^{2}+\left(a\lambda k-t_{1}\right)^{2}\right] \\
&= \frac{1}{2}\left[\frac{\lambda}{1+a^{2}\lambda}\sigma_{e}^{2}+\lambda\left(k\right)^{2}+\left(a\lambda k-a\lambda k\right)^{2}\right] \\
&= \frac{1}{2}\frac{\lambda}{1+a^{2}\lambda}\sigma_{e}^{2}+\frac{\lambda}{2}\left(k\right)^{2}
\end{aligned}$$

which is equal to (18).